Smooth $U(1)$ Gauge Potentials on the de Sitter Spacetime

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Abstract

1 Introduction

Classical electrodynamics, most prominently developed by James Clerk Maxwell, is one of the greatest theories ever since Newton's mechanics. In classical electrodynamics, there are some key points concerned in this paper:

(1) The existence of electromagnetic waves was predicted theoretically, then confirmed by experiments by Heinrich Hertz;

(2) Electromagnetic fields carry energy and momentum as perfect as ordinary matters. Neither the energy-momentum tensor of ordinary matters nor that of electromagnetic fields is conserved. Instead, the conserved energy-momentum tensor must be the sum for both that of ordinary matters and that of electromagnetic fields.

(3) There are nontrivial solutions of source-free Maxwell equations, meaning that electromagnetic fields could exist independent of charges and currents.

(4) In this way electromagnetic fields are recognized as an existing form of matters, not just an imaginary or mathematical concept.

(5) In vacuum, electromagnetic waves propagate at a universal speed c, having nothing to do with time, space point, propagation direction, as well as the state of the emitting source. The speed c is that of light in vacuum. Thus James C. Maxwell concluded that light is nothing but a kind of electromagnetic waves, hence dominated by the Maxwell equations.

(6) It also seems that the speed of light, in vacuum, is independent of (inertial) reference frames.

The validity of point (6) was an open question before 1905. This question, together with the problem of the covariance of electrodynamics, finally resulted in the special theory of relativity (SR), and then the general theory of relativity (GR). Especially, point (6) is recognized by Albert Einstein as a fundamental postulate of SR, the principle of invariant light speed.

In SR, electrodynamics is well adapted to the relativistic concept of spacetime, with all the above points perfectly preserved.

In GR, most of dynamical equations (except Einstein's field equation and so on) are modified from those in SR. For long time it is thought that consequence of relativistic dynamics remains still valid in GR, at least remained in a modified form. For examples, points (1) to (4) in the above are thought to be still correct in GR, while points (5) and (6) are thought to be valid in a modified form.

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Speed of light in vacuum. In general relativity, giving solutions of sourceless electromagnetic waves. Physical laws.

In this paper, we show, by virtue of an example, that it is not the case: basic knowledge obtained from SR is not necessarily valid in GR. To be specific, we shall show that there should be no source-free electromagnetic fields in the de Sitter background. In SR, the distribution of (electrical/magnetic) charges and currents cannot determine the electromagnetic fields. To determine the latter, initial condition and boundary conditions are needed. In the de Sitter background, however, initial condition and boundary conditions are not necessary; the distribution of (electric/magnetic) charges and currents is enough to determine the electromagnetic field.

The absence of source-free electromagnetic fields in de Sitter background brings out a serious problem: the experimental foundation of the curved metric. In GR, the numerical value of light speed might somehow be meaningless. But the 4-"velocity" of a photon in vacuum makes sense, which is specified as a lightlike vector with respect to the metric, and vice versa. By means of lightlike vectors of various directions, the spacetime metric can be determined up to a conformal factor. This is the experimental foundation of the curved metric in GR. If there is no source-free electromagnetic fields, there is no way to obtain lightlike 4-vectors. In this case the curved metric will lose its experimental foundation.

A-J relation.

In this paper we are not going to discuss such a problem. We mainly focus on whether there is source-free electromagnetic fields in the de Sitter background. The fundamental technique to solve this problem is the application of theory of Lie groups and Lie algebras.

This paper is organized as the following. In section 2 we outline the ideas and present some preliminary formulae. In section 3 vector fields, as the highest weight vector in the representation of the Lie algebra $\mathfrak{so}(1,4)$, are obtained. Following the standard method, Verma modules of $\mathfrak{so}(1,4)$, consisting of vector fields, or 1-forms, or 3-forms, respectively, are obtained. These Verma modules are irreducible $\mathfrak{so}(1,4)$-modules. They are described in section 4. In section 5 some important properties of 1-forms in the above modules are discussed. These are applicable in constructing smooth solutions of the Maxwell equations on dS^4, or smooth solutions of the Proc equation. These solutions are described in section 6 and section 7, respectively. In section 8 we summarize the whole paper, and discussed some related problems.

$\Omega^1(dS^4)_\infty$. Related work. Technique. Embedding not necessary. Organization.

2 Some Preliminary Formulae

The coordinate system $(\chi,\zeta,\theta,\varphi)$ on dS^4 is defined by

$$\xi^0 = l\sinh\chi\,, \tag{1}$$
$$\xi^1 = l\cosh\chi\cos\zeta\cos\theta\,, \tag{2}$$
$$\xi^2 = l\cosh\chi\cos\zeta\sin\theta\,, \tag{3}$$
$$\xi^3 = l\cosh\chi\sin\zeta\cos\varphi\,, \tag{4}$$
$$\xi^4 = l\cosh\chi\sin\zeta\sin\varphi\,. \tag{5}$$

3 Highest Weight Vector Fields

We first start to find a vector field

$$\mathbf{v}_\lambda = v^0\frac{\partial}{\partial\chi} + v^1\frac{\partial}{\partial\zeta} + v^2\frac{\partial}{\partial\theta} + v^3\frac{\partial}{\partial\varphi} \tag{6}$$

on the de Sitter spacetime, acting as a highest weight vector in the representation of $\mathfrak{so}(1,4)$ with the weight $\lambda = N_1\lambda_1 + N_2\lambda_2$. The action of $\mathfrak{so}(1,4)$ on vector fields is realized by Lie derivatives. Then $\mathbf{v}_\lambda$ satisfies

$$L_{\mathbf{h}_{\alpha_1}}\mathbf{v}_\lambda = [\mathbf{h}_{\alpha_1},\mathbf{v}_\lambda] = N_1\mathbf{v}_\lambda\,, \qquad L_{\mathbf{h}_{\alpha_2}}\mathbf{v}_\lambda = [\mathbf{h}_{\alpha_2},\mathbf{v}_\lambda] = N_2\mathbf{v}_\lambda\,, \tag{7}$$
$$L_{\mathbf{e}_{\alpha_1}}\mathbf{v}_\lambda = [\mathbf{e}_{\alpha_1},\mathbf{v}_\lambda] = 0\,, \qquad L_{\mathbf{e}_{\alpha_1+2\alpha_2}}\mathbf{v}_\lambda = [\mathbf{e}_{\alpha_1+2\alpha_2},\mathbf{v}_\lambda] = 0\,, \tag{8}$$
$$L_{\mathbf{e}_{\alpha_2}}\mathbf{v}_\lambda = [\mathbf{e}_{\alpha_2},\mathbf{v}_\lambda] = 0\,, \qquad L_{\mathbf{e}_{\alpha_1+\alpha_2}}\mathbf{v}_\lambda = [\mathbf{e}_{\alpha_1+\alpha_2},\mathbf{v}_\lambda] = 0\,. \tag{9}$$

Eqs. (7) result in

$$i\,\frac{\partial v^\mu}{\partial\theta} - i\,\frac{\partial v^\mu}{\partial\varphi} = N_1\,v^\mu\,, \qquad 2i\,\frac{\partial v^\mu}{\partial\varphi} = N_2\,v^\mu\,.$$

The general solution of the above equations is

$$v^\mu = V^\mu(\chi,\zeta)\,e^{-i(N_1+\frac{N_2}{2}\theta)-i\frac{N_2}{2}\varphi}\,,$$

where $V^\mu(\chi,\zeta)$ are some functions depending only on χ and ζ. Hence

$$\mathbf{v}_\lambda = e^{-i(N_1+\frac{N_2}{2})\theta - i\frac{N_2}{2}\varphi}\Big(V^0(\chi,\zeta)\,\frac{\partial}{\partial\chi} + V^1(\chi,\zeta)\,\frac{\partial}{\partial\zeta} + V^2(\chi,\zeta)\,\frac{\partial}{\partial\theta} + V^3(\chi,\zeta)\,\frac{\partial}{\partial\varphi}\Big)\,. \tag{10}$$

According to eqs. (1) to (5), the spacetime point returns back to its initial position whenever φ increases by 2π. The above expression implies that N_2 must be an even integer. Soon we shall see that N_2 could be either zero or 2.

In fact, we can substitute the above expression into eqs. (8), obtaining eight equations for four unknown functions $V^\mu(\chi,\zeta)$. Among these equations, there are four PDEs and four linear algebraic equations. All these equations force N_2 to be either ±2 or 0.

For $N_2=0$, these equations are reduced to

$$\begin{aligned}
&\frac{\partial V^0}{\partial\zeta} + N_1\,V^0\tan\zeta = 0\,,\\
&\frac{\partial V^2}{\partial\zeta} + (N_1-2)\,V^2\tan\zeta = 0\,,
\end{aligned}$$

together with

$$V^1 = -iV^2\sin\zeta\cos\zeta\,,\qquad V^3 = 0\,.$$

The general solution of these equations can be easily obtained, resulting in the corresponding vector field $\mathbf{v}_\lambda$ to be

$$\mathbf{v}_{N\lambda_1} = e^{-iN\theta}\cos^N\zeta\,\Big(\mathcal{V}^0\,\frac{\partial}{\partial\chi} - i\mathcal{V}^2\tan\zeta\,\frac{\partial}{\partial\zeta} + \mathcal{V}^2\sec^2\zeta\,\frac{\partial}{\partial\theta}\Big)\,,$$

where N_1 has been denoted simply by N. Substitution of this expression into the first equation in (9) yields

$$\begin{aligned}
&\frac{\mathrm{d}\mathcal{V}^2}{\mathrm{d}\chi} - (N-2)\,\mathcal{V}^2\tanh\chi = 0\,,\\
&\mathcal{V}^0 = -i\mathcal{V}^2\,\sinh\chi\,\cosh\chi\,.
\end{aligned} \tag{11}$$

The general solution of the above equations is

$$\mathcal{V}^2 = C\,(\cosh\chi)^{N-2}\,,\qquad \mathcal{V}^0 = -iC\,\sinh\chi\,(\cosh\chi)^{N-1}\,,$$

where C is the integral constant. We can fix C as iN/l (where l is the cosmological radius of dS^4) so that

$$\mathbf{v}_{N\lambda_1} = \frac{N}{l}\,\phi_{N\lambda_1}\Big(\tanh\chi\,\frac{\partial}{\partial\chi} + \frac{\tan\zeta}{\cosh^2\chi}\,\frac{\partial}{\partial\zeta} + i\frac{\sec^2\zeta}{\cosh^2}\,\frac{\partial}{\partial\theta}\Big)\,, \tag{12}$$

which satisfies

$$g_{ab}v^b_{N\lambda_1} = l\,(\mathrm{d}\phi_{N\lambda_1})_a\,. \tag{13}$$

Here

$$\phi_{N\lambda_1} = \phi^N_{\lambda_1} = (e^{-i\theta}\cosh\chi\cos\zeta)^N \tag{14}$$

is the highest weight scalar with the weight $N\lambda_1$. For details of this function, we refer to [1]. Since the function $\phi_{N\lambda_1}$ is smooth on dS^4 [1], so is the 1-form $\mathrm{d}\phi_{N\lambda_1}$. As a consequence, $\mathbf{v}_{N\lambda_1}$ is a smooth vector field on dS^4.

It can be verified that the second equation in (9) is automatically satisfied.

For the case of $N_2\neq0$, those eight equations forces $V^0=0$. In order that one of V^1, V^2 and V^3 is nonzero in these equations, N_2 must be ±2. But according to the representation theory, $\lambda = N_1\lambda_1 + N_2\lambda_2$ is a dominant weight, so that $N_2=2$.[1] Then those equations, derived from eqs. (8), are reduced to

$$\begin{aligned}
&V^0 = 0\,,\\
&\frac{\partial V^1}{\partial\zeta} + N_1\,V^1\tan\zeta = 0\,,\\
&V^2 = iV^1\tan\zeta\,,\\
&V^3 = -iV^1\cot\zeta\,.
\end{aligned}$$

[1] When $N_2=-2$, one can try to solve the equations. Then only a zero 1-form could be obtained.

Having easily obtained the general solutions of these equations, there is the corresponding

$$\begin{aligned}\mathbf{v}_\lambda &= e^{-i(N+1)\theta-i\varphi}\,\mathcal{V}(\chi)\,(\cos\zeta)^N\Big(\frac{\partial}{\partial\zeta}+i\tan\zeta\,\frac{\partial}{\partial\theta}-i\cot\zeta\,\frac{\partial}{\partial\varphi}\Big)\\ &= 2\mathcal{V}(\chi)\,(e^{-i\theta}\cos\zeta)^N\,\mathbf{e}_{\alpha_1+2\alpha_2}\end{aligned}\tag{15}$$

with the highest weight $\lambda = N\lambda_1 + 2\lambda_2$. In this expression, $\mathcal{V}$ is a function depending on χ only. For the expression of $\mathbf{e}_{\alpha_1+2\alpha_2}$, we refer to [1]. It can be calulated that

$$L_{\mathbf{e}_{\alpha_2}}\mathbf{v}_\lambda = 2e^{-iN\theta-i\varphi}\Big(\frac{\mathrm{d}\mathcal{V}}{\mathrm{d}\chi}-N\mathcal{V}\tanh\chi\Big)\sin\zeta\,(\cos\zeta)^N\,\mathbf{e}_{\alpha_1+2\alpha_2}\,.$$

Therefore the first equation in (9) requires that

$$\frac{\mathrm{d}\mathcal{V}}{\mathrm{d}\chi}-N\mathcal{V}\tanh\chi = 0\,,$$

which has the general solution $\mathcal{V} = C(\cosh\chi)^N$. With the integral constant C fixed as $1/l$, we can obtain a highest weight vector field

$$\mathbf{v}_\lambda = \frac{2}{l}\,\phi_{N\lambda_1}\,\mathbf{e}_{\alpha_1+2\alpha_2}\,,\qquad (\lambda = N\lambda_1+2\lambda_2)\,.\tag{16}$$

Now that both $\phi_{N\lambda_1}$ and $\mathbf{e}_{\alpha_1+2\alpha_2}$ are smooth on dS^4 [1], obviously $\mathbf{v}_\lambda$ is a smooth vector field on dS^4.

By virtue of eq. (16) as well as the relations in [1], it is obvious that the second equation in (9) is automatically satisfied.

4 Verma Modules of Smooth Vector Fields, 1-Forms and/or 3-Forms

As we have seen in the above, for an irreducible $\mathfrak{so}(1,4)$ module of smooth vector fields on dS^4, its highest weight λ is either $N\lambda_1$ or $N\lambda_1+2\lambda_2$, with N a non-negative integer. Given such a highest weight λ, the corresponding Verma module of smooth vector fields on dS^4 will be denoted by $\mathfrak{X}(dS^4)_\lambda$, which is spanned by the following vector fields:

$$\mathbf{v}_\lambda^{(jklm)} := L^j_{\mathbf{f}_{\alpha_1+\alpha_2}}L^k_{\mathbf{f}_{\alpha_1+2\alpha_2}}L^l_{\mathbf{f}_{\alpha_1}}L^m_{\mathbf{f}_{\alpha_2}}\mathbf{v}_\lambda\tag{17}$$

for some non-negative integer j, k, l and m. Here, the notation $L^j_{\mathbf{f}_{\alpha_2}}$ stands for the action of the Lie derivative $L_{\mathbf{f}_{\alpha_2}}$ for j times (provided that $j>0$), while $L^0_{\mathbf{f}_{\alpha_2}}$ stands for the identity map, and so on. In this manner each $\mathbf{v}\in\mathfrak{X}(dS^4)_\lambda$ can be written as a linear combination

$$\mathbf{v} = \sum_{j=0}^{j_{\max}}\sum_{k=0}^{k_{\max}}\sum_{l=0}^{l_{\max}}\sum_{m=0}^{m_{\max}}C_{jklm}\mathbf{v}_\lambda^{(jklm)}\tag{18}$$

for some constants C_{jklm}, where the integers $j_{\max}$, $k_{\max}$, $l_{\max}$ and $m_{\max}$ can be determined by the knowledge of the weight diagram. This will be described later.

Via the Lie derivatives, the Lie algebra $\mathfrak{so}(1,4)$ also acts on $\Omega^p(dS^4)$, the infinite dimensional vector space of p-forms on dS^4. Given a dominant weight $\lambda = N_1\lambda_1+N_2\lambda_2$, there might be a p-form $\boldsymbol{\alpha}_\lambda$ satisfying

$$L_{\mathbf{h}_{\alpha_1}}\boldsymbol{\alpha}_\lambda = N_1\boldsymbol{\alpha}_\lambda\,,\qquad L_{\mathbf{h}_{\alpha_2}}\boldsymbol{\alpha}_\lambda = N_2\boldsymbol{\alpha}_\lambda\,,\tag{19}$$

$$L_{\mathbf{e}_{\alpha_1}}\boldsymbol{\alpha}_\lambda = 0\,,\qquad L_{\mathbf{e}_{\alpha_1+2\alpha_2}}\boldsymbol{\alpha}_\lambda = 0\,,\tag{20}$$

$$L_{\mathbf{e}_{\alpha_2}}\boldsymbol{\alpha}_\lambda = 0\,,\qquad L_{\mathbf{e}_{\alpha_1+\alpha_2}}\boldsymbol{\alpha}_\lambda = 0\,.\tag{21}$$

Were there such a nonzero p-form $\boldsymbol{\alpha}_\lambda$, there will be a corresponding Verma module $\Omega^p(dS^4)_\lambda$ with the highest weight λ, spanned by the p-forms such like

$$\boldsymbol{\alpha}_\lambda^{(jklm)} := L^j_{\mathbf{f}_{\alpha_1+\alpha_2}}L^k_{\mathbf{f}_{\alpha_1+2\alpha_2}}L^l_{\mathbf{f}_{\alpha_1}}L^m_{\mathbf{f}_{\alpha_2}}\boldsymbol{\alpha}_\lambda\,,\tag{22}$$

hence a p-form $\boldsymbol{\alpha}\in\Omega^p(dS^4)_\lambda$ can be expressed as

$$\boldsymbol{\alpha} = \sum_{j=0}^{j_{\max}}\sum_{k=0}^{k_{\max}}\sum_{l=0}^{l_{\max}}\sum_{m=0}^{m_{\max}}C_{jklm}\boldsymbol{\alpha}_\lambda^{(jklm)}\tag{23}$$

for some constants C_{jklm}. Similarly, the integers $j_{\max}$, $k_{\max}$, $l_{\max}$ and $m_{\max}$ can be determined by the knowledge of the weight diagram.

For $p = 0$, $\Omega^0(dS^4)$ and $\Omega^0(dS^4)_\lambda$ are identified with $C^\infty(dS^4)$ and $C^\infty(dS^4)_\lambda$, respectively. According to the study in [1], possible λ for the existence of $C^\infty(dS^4)_\lambda$ could be merely $\lambda = N\lambda_1$.

Since the Hodge $*$-operator is determined by the orientation and the metric on dS^4, it is straightforward that the Lie derivative with respect to a Killing vector field $\mathbf{X}$ is commutative with $*$:

$$* L_{\mathbf{X}} = L_{\mathbf{X}} * \tag{24}$$

As a consequence, a p-form $\boldsymbol{\alpha}_\lambda$ satisfying eqs. (19) to (21) generates a $(4-p)$-form

$$\boldsymbol{\alpha}'_\lambda := *\boldsymbol{\alpha}_\lambda\,, \tag{25}$$

which also satisfies a set of equations similar to eqs. (19) to (21). Therefore, when the Verma module $\Omega^p(dS^4)_\lambda$ exists, there is also a Verma module $\Omega^{4-p}(dS^4)_\lambda = *\Omega^p(dS^4)_\lambda$ with the same highest weight λ. What's more, these two Verma modules are isomorphic to each other, which can be seen from

$$\boldsymbol{\alpha}'^{(jklm)}_\lambda = *\boldsymbol{\alpha}^{(jklm)}_\lambda\,. \tag{26}$$

Here $\boldsymbol{\alpha}'^{(jklm)}_\lambda$ is defined by

$$\boldsymbol{\alpha}'^{(jklm)}_\lambda := L^j_{\mathbf{f}_{\alpha_1+\alpha_2}} L^k_{\mathbf{f}_{\alpha_1+2\alpha_2}} L^l_{\mathbf{f}_{\alpha_1}} L^m_{\mathbf{f}_{\alpha_2}} \boldsymbol{\alpha}'_\lambda\,. \tag{27}$$

For convenience, the map sending a vector field $\mathbf{v}$ to a 1-form $\tilde{\mathbf{v}}$, or v^a to $v_a = g_{ab}v^b$ in the convention of abstract indices, is denoted by $\mathbf{g}_\flat\colon \mathfrak{X}(dS^4) \to \Omega^1(dS^4)$ in this paper. Then, for each Killing vector field $\mathbf{X}$, there is $L_{\mathbf{X}} v_a = g_{ab}\, L_{\mathbf{X}} v^b$ for arbitrary vector field v^a, namely,

$$L_{\mathbf{X}}\mathbf{g}_\flat = \mathbf{g}_\flat L_{\mathbf{X}}\,. \tag{28}$$

This indicates that $\mathbf{g}_\flat$ is an isomorphism of $\mathfrak{so}(1,4)$-modules: with respect to the actions of $\mathfrak{so}(1,4)$, each property of $\mathfrak{X}(dS^4)$ is precisely mapped to $\Omega^1(dS^4)$. Especially, the Verma module $\mathfrak{X}(dS^4)_\lambda$ is mapped by $\mathbf{g}_\flat$ to the Verma module $\Omega^1(dS^4)_\lambda$. As a consequence, possible highest weights for $\Omega^1(dS^4)_\lambda$ are either $N\lambda_1$ or $N\lambda_1 + 2\lambda_2$, with N a non-negative integer. Then, by the Hodge $*$-operator, these are also the highest weights of $\Omega^3(dS^4)_\lambda$.

For the highest weight $\lambda = N\lambda_1$, it is the most convenient to investigate $\Omega^1(dS^4)_{N\lambda_1}$, instead of $\mathfrak{X}(dS^4)_{N\lambda_1}$ and $\Omega^3(dS^4)_{N\lambda_1}$, because the highest weight 1-form

$$\tilde{\mathbf{v}}_{N\lambda_1} = \mathbf{g}_\flat \mathbf{v}_{N\lambda_1} = l\, \mathrm{d}\phi_{N\lambda_1}\,, \tag{29}$$

as shown in eq. (13). Then, due to $L_{\mathbf{X}}\mathrm{d} = \mathrm{d}L_{\mathbf{X}}$ for an arbitrary vector field $\mathbf{X}$,

$$\tilde{\mathbf{v}}^{(jklm)}_{N\lambda_1} = L^j_{\mathbf{f}_{\alpha_1+\alpha_2}} L^k_{\mathbf{f}_{\alpha_1+2\alpha_2}} L^l_{\mathbf{f}_{\alpha_1}} L^m_{\mathbf{f}_{\alpha_2}} \tilde{\mathbf{v}}_{N\lambda_1} = l\, \mathrm{d}(L^j_{\mathbf{f}_{\alpha_1+\alpha_2}} L^k_{\mathbf{f}_{\alpha_1+2\alpha_2}} L^l_{\mathbf{f}_{\alpha_1}} L^m_{\mathbf{f}_{\alpha_2}} \phi_{N\lambda_1})\,.$$

Because of $L_{\mathbf{f}_{\alpha_2}}\phi_{N\lambda_1} = 0$ (see, [1]), there is $\tilde{\mathbf{v}}^{(jklm)}_{N\lambda_1} = 0$ whenever $m > 0$. So, nonzero $\tilde{\mathbf{v}}^{(jklm)}_{N\lambda_1}$ is among those with $m = 0$:

$$\tilde{\mathbf{v}}^{(jkl0)}_{N\lambda_1} = l\, \mathrm{d}\phi^{(jkl)}_{N\lambda_1}\,. \tag{30}$$

For the expression of $\phi^{(jkl)}_{N\lambda_1}$, we refer to [1]. Thus the Verma module

$$\Omega^1(dS^4)_{N\lambda_1} = \mathrm{d}[\Omega^0(dS^4)_{N\lambda_1}] = \mathrm{d}[C^\infty(dS^4)_{N\lambda_1}]\,, \tag{31}$$

consisting of exact 1-forms only.

The Verma modules $\mathfrak{X}(dS^4)_{N\lambda_1}$, $\Omega^1(dS^4)_{N\lambda_1}$ and $\Omega^3(dS^4)_{N\lambda_1}$ are all irreducible $\mathfrak{so}(1,4)\otimes\mathbb{C}$-modules.

For the highest weight $\lambda = N\lambda_1 + 2\lambda_2$, it is the most convenient to investigate $\mathfrak{X}(dS^4)_\lambda$, instead of $\Omega^1(dS^4)_\lambda$ and $\Omega^3(dS^4)_\lambda$. This module is spanned by vector fields like those in eq. (17), for which $\mathbf{v}_\lambda$ is as shown in eq. (16). Their

expressions are listed below:

$$\begin{aligned}
\mathbf{v}_\lambda^{(jk\ell 0)} &= \frac{2}{l}\phi_{N\lambda_1}^{(jk\ell)}\,\mathbf{e}_{\alpha_1+2\alpha_2} - \frac{2j}{l}\phi_{N\lambda_1}^{(j-1,k\ell)}\,\mathbf{e}_{\alpha_2} - \frac{2k}{l}\phi_{N\lambda_1}^{(j,k-1,\ell)}\,(\mathbf{h}_{\alpha_1}+\mathbf{h}_{\alpha_2}) + \frac{2j(j-1)}{l}\phi_{N\lambda_1}^{(j-2,k\ell)}\,\mathbf{f}_{\alpha_1} \\
&\quad - \frac{2jk}{l}\phi_{N\lambda_1}^{(j-1,k-1,\ell)}\,\mathbf{f}_{\alpha_1+\alpha_2} - \frac{2k\,(k-1)}{l}\phi_{N\lambda_1}^{(j,k-2,\ell)}\,\mathbf{f}_{\alpha_1+2\alpha_2}\,, \qquad (32)\\
\mathbf{v}_\lambda^{(jkl1)} &= \frac{2}{l}\,\phi_{N\lambda_1}^{(jk\ell)}\,\mathbf{e}_{\alpha_1+\alpha_2} + \frac{2\ell}{l}\,\phi_{N\lambda_1}^{(jk,\ell-1)}\,\mathbf{e}_{\alpha_2} - \frac{2j}{l}\,\phi_{N\lambda_1}^{(j-1,k\ell)}\,(2\,\mathbf{h}_{\alpha_1}+\mathbf{h}_{\alpha_2}) - \frac{4j\ell}{l}\,\phi_{N\lambda_1}^{(j-1,k,\ell-1)}\,\mathbf{f}_{\alpha_1} - \frac{2k}{l}\,\phi_{N\lambda_1}^{(j,k-1,\ell)}\,\mathbf{f}_{\alpha_2} \\
&\quad + \frac{2}{l}\,(k\ell\,\phi_{N\lambda_1}^{(j,k-1,\ell-1)} - j(j-1)\,\phi_{N\lambda_1}^{(j-2,k\ell)})\,\mathbf{f}_{\alpha_1+\alpha_2} - \frac{4jk}{l}\,\phi_{N\lambda_1}^{(j-1,k-1,\ell)}\,\mathbf{f}_{\alpha_1+2\alpha_2}\,. \qquad (33)\\
\mathbf{v}_\lambda^{(jk\ell 2)} &= -\frac{4}{l}\,\phi_{N\lambda_1}^{(jk\ell)}\,\mathbf{e}_{\alpha_1} + \frac{4\ell}{l}\,\phi_{N\lambda_1}^{(jk,\ell-1)}\,\mathbf{h}_{\alpha_1} + \frac{4\ell(\ell-1)}{l}\,\phi_{N\lambda_1}^{(jk,\ell-2)}\,\mathbf{f}_{\alpha_1} - \frac{4j}{l}\,\phi_{N\lambda_1}^{(j-1,k\ell)}\,\mathbf{f}_{\alpha_2} \\
&\quad + \frac{4j\ell}{l}\,\phi_{N\lambda_1}^{(j-1,k,\ell-1)}\,\mathbf{f}_{\alpha_1+\alpha_2} - \frac{4j(j-1)}{l}\,\phi_{N\lambda_1}^{(j-2,k\ell)}\,\mathbf{f}_{\alpha_1+2\alpha_2}\,, \qquad (34)\\
\mathbf{v}_\lambda^{(jk\ell m)} &= 0\,, \qquad (m>2)\,. \qquad (35)
\end{aligned}$$

5 Properties of 1-Forms in Each Irreducible $\mathfrak{so}(1,4)$-Module

For a vector field $\mathbf{v}$, the 1-form $v_a = g_{ab}v^b$ is also denoted by $\tilde{\mathbf{v}}$ without abstract indices, in this paper. For a dominant weight $\lambda = N\lambda_1 + 2\lambda_2$, the corresponding highest weight vector field $\mathbf{v}_\lambda$, as shown in eq. (16), has

$$\tilde{\mathbf{v}}_\lambda = \frac{2}{l}\phi_{N\lambda_1}\,\tilde{\mathbf{e}}_{\alpha_1+2\alpha_2} = -l\,(\cosh\chi)^{N+2}(\cos\zeta)^N e^{-i(N+1)\theta-i\varphi}[\mathrm{d}\zeta + i\sin\zeta\cos\zeta\,(\mathrm{d}\theta-\mathrm{d}\varphi)]\,. \qquad (36)$$

It is straightforward to obtain that

$$\begin{aligned}
\mathrm{d}\tilde{\mathbf{v}}_\lambda &= -(N+2)l\phi_{\lambda_1}^N e^{-i\theta-i\varphi}\sinh\chi\cosh\chi\,\mathrm{d}\chi\wedge[\mathrm{d}\zeta + i\sin\zeta\cos\zeta\,(\mathrm{d}\theta-\mathrm{d}\varphi)] \\
&\quad -(N+2)l\phi_{\lambda_1}^N e^{-i\theta-i\varphi}\cosh^2\chi\,(i\cos^2\zeta\,\mathrm{d}\zeta\wedge\mathrm{d}\theta + i\sin^2\zeta\,\mathrm{d}\zeta\wedge\mathrm{d}\varphi - \sin\zeta\cos\zeta\,\mathrm{d}\theta\wedge\mathrm{d}\varphi)\,. \qquad (37)
\end{aligned}$$

The coordinate system $(\chi,\zeta,\theta,\varphi)$ is not always compatible with the orientation of dS^4. That means, the volume 4-form is not necessarily $\sqrt{|g|}\,\mathrm{d}\chi\wedge\mathrm{d}\zeta\wedge\mathrm{d}\theta\wedge\mathrm{d}\varphi$: sometimes it could differ by a negative sign. To obtain the correct expression of the volume 4-form $\boldsymbol{\varepsilon}$ on dS^4, we notice that on $\mathbb{R}^{1,4}$ there is the standard volume 5-form $\overline{\boldsymbol{\varepsilon}} = \mathrm{d}\xi^0\wedge\mathrm{d}\xi^1\wedge\cdots\wedge\mathrm{d}\xi^4$. Then, setting

$$\mathbf{D} = \xi^A\frac{\partial}{\partial\xi^A}\,, \qquad (38)$$

the volume 4-form $\boldsymbol{\varepsilon}$ on dS^4 is just the pull-back of $\frac{1}{l}\,i_{\mathbf{D}}\overline{\boldsymbol{\varepsilon}}$ to the hypersurface dS^4 in $\mathbb{R}^{1,4}$. In terms of the coordinates χ, ζ, θ and φ,

$$\boldsymbol{\varepsilon} = l^4\cosh^3\chi\sin\zeta\cos\zeta\;\mathrm{d}\chi\wedge\mathrm{d}\zeta\wedge\mathrm{d}\theta\wedge\mathrm{d}\varphi\,. \qquad (39)$$

For the Hodge dual,

$$\begin{aligned}
*\,(\mathrm{d}\chi\wedge\mathrm{d}\zeta) &= -\cosh\chi\sin\zeta\cos\zeta\,\mathrm{d}\theta\wedge\mathrm{d}\varphi\,, &(40)\\
*\,(\mathrm{d}\chi\wedge\mathrm{d}\theta) &= \cosh\chi\tan\zeta\,\mathrm{d}\zeta\wedge\mathrm{d}\varphi\,, &(41)\\
*\,(\mathrm{d}\chi\wedge\mathrm{d}\varphi) &= -\cosh\chi\cot\zeta\,\mathrm{d}\zeta\wedge\mathrm{d}\theta\,, &(42)\\
*\,(\mathrm{d}\zeta\wedge\mathrm{d}\theta) &= \frac{\tan\zeta}{\cosh\chi}\,\mathrm{d}\chi\wedge\mathrm{d}\varphi\,, &(43)\\
*\,(\mathrm{d}\zeta\wedge\mathrm{d}\varphi) &= -\frac{\cot\zeta}{\cosh\chi}\,\mathrm{d}\chi\wedge\mathrm{d}\theta\,, &(44)\\
*\,(\mathrm{d}\theta\wedge\mathrm{d}\varphi) &= \frac{\sec\zeta\csc\zeta}{\cosh\chi}\,\mathrm{d}\chi\wedge\mathrm{d}\zeta\,. &(45)
\end{aligned}$$

By virtue of these relations, one has

$$*\,\mathrm{d}*\mathrm{d}\tilde{\mathbf{v}}_\lambda = -\frac{(N+2)(N+3)}{l^2}\,\tilde{\mathbf{v}}_\lambda\,. \qquad (46)$$

Consequently,

$$*\;\mathrm{d}*\tilde{\mathbf{v}}_\lambda = 0\,. \qquad (47)$$

Since both $*$ and d commute with the Lie derivatives, one has

$$*\,\mathrm{d}*\mathrm{d}\tilde{\mathbf{v}}_{\lambda}^{(jk\ell m)} = -\frac{(N+2)(N+3)}{l^2}\tilde{\mathbf{v}}_{\lambda}^{(jk\ell m)}\,, \tag{48}$$

$$*\,\mathrm{d}*\tilde{\mathbf{v}}_{\lambda}^{(jk\ell m)} = 0\,, \tag{49}$$

for arbitrary nonnegative integers j, k, ℓ and m. Hence every 1-form $\tilde{\mathbf{v}} \in \Omega^1(dS^4)_\lambda$, with $\lambda = N\lambda_1 + 2\lambda_2$, satisfies

$$*\,\mathrm{d}*\mathrm{d}\tilde{\mathbf{v}} = -\frac{(N+2)(N+3)}{l^2}\tilde{\mathbf{v}}\,, \tag{50}$$

$$*\,\mathrm{d}*\tilde{\mathbf{v}} = 0\,. \tag{51}$$

For a dominant weight $\lambda = N\lambda_1$, the corresponding highest weight vector field $\mathbf{v}_{N\lambda_1}$ has

$$\tilde{\mathbf{v}}_{N\lambda_1} = \mathrm{d}\,(l\phi_{N\lambda_1}) = Nl\,\phi_{\lambda_1}^{N-1}\,\mathrm{d}\phi_{\lambda_1}\,, \tag{52}$$

as indicated in eq. (13). Then, due to the identities

$$-\nabla_a v^a_{N\lambda_1} = *\,\mathrm{d}*\tilde{\mathbf{v}}_{N\lambda_1} = *\,\mathrm{d}*\mathrm{d}(l\phi_{N\lambda_1}) = -g^{ab}\nabla_a\nabla_b(l\phi_{N\lambda_1}) \tag{53}$$

and the result in [1], there will be

$$*\;\mathrm{d}*\tilde{\mathbf{v}}_{N\lambda_1} = -\frac{N(N+3)}{l}\phi_{N\lambda_1} = -\nabla_a v^a_{N\lambda_1}\,. \tag{54}$$

Note that $\tilde{\mathbf{v}}_{N\lambda_1}$ is zero when $N=0$. Hence the $\nabla_a v^a_{N\lambda_1}$ is nonzero whenever $\tilde{\mathbf{v}}_{N\lambda_1}$ is nonzero. It follows that, for each $\tilde{\mathbf{v}} \in \Omega^1(dS^4)_{N\lambda_1}$, there is a smooth function ϕ on dS^4 satisfying

$$\tilde{\mathbf{v}} = \mathrm{d}\phi\,, \qquad *\,\mathrm{d}*\tilde{\mathbf{v}} = -\nabla_a v^a = -\frac{N(N+3)}{l}\phi\,. \tag{55}$$

In fact, such a function ϕ could be chosen from the irreducible $\mathfrak{so}(1,4)$-module $C^\infty(dS^4)_{N\lambda_1}$.

$$g_{ab}v^a_{N\lambda_1+2\lambda_2}v^b_{N\lambda_1+2\lambda_2} = \frac{4}{l^2}\,\phi_{2N\lambda_1}g_{ab}\,e^a_{\alpha_1+2\alpha_2}e^b_{\alpha_1+2\alpha_2} \tag{56}$$

6 Smooth Solutions of the Maxwell Equations on dS^4

Now that we have obtained all finite dimensional irreducible $\mathfrak{so}(1,4)$-submodules of $\Omega^1(dS^4)$, the space of smooth 1-forms on dS^4, there is the direct sum decomposition

$$\Omega^1(dS^4) = \Omega^1(dS^4)_{\text{fin}} \oplus \Omega^1(dS^4)_\infty\,, \tag{57}$$

$$\Omega^1(dS^4)_{\text{fin}} := \bigoplus_{N=0}^{\infty}\Omega^1(dS^4)_{N\lambda_1} \oplus \bigoplus_{N=0}^{\infty}\Omega^1(dS^4)_{N\lambda_1+2\lambda_2}\,. \tag{58}$$

Here $\Omega^1(dS^4)_\infty$ is the direct of all infinite dimensional irreducible $\mathfrak{so}(1,4)$-submodules of $\Omega^1(dS^4)$ or just the zero subspace, depending on the existence of infinite dimensional irreducible $\mathfrak{so}(1,4)$-submodules in $\Omega^1(dS^4)$.

It remains an open problem whether $\Omega^1(dS^4)$ has an infinite dimensional irreducible $\mathfrak{so}(1,4)$-submodule. Currently we discuss smooth solutions the Maxwell equations with the work assumption that $\Omega^1(dS^4)_\infty = 0$. In other words, when we talk about a smooth 1-form (or, equivalently, a smooth vector field), we are referring to one contained in the direct sum $\Omega^1(dS^4)_{\text{fin}}$.

Thus a 4-potential $\mathbf{A} = A_\mu\,\mathrm{d}x^\mu$ can be uniquely decomposed into

$$\mathbf{A} = \sum_N(\mathbf{A}_{N\lambda_1} + \mathbf{A}_{N\lambda_1+2\lambda_2}) \tag{59}$$

with $\mathbf{A}_{N\lambda_1} \in \Omega^1(dS^4)_{N\lambda_1}$ and $\mathbf{A}_{N\lambda_1+2\lambda_2} \in \Omega^1(dS^4)_{N\lambda_1+2\lambda_2}$. If one requires the 4-potential $\mathbf{A}$ to satisfy the Lorentz gauge condition

$$-\nabla_a A^a = *\,\mathrm{d}*\mathbf{A} = 0\,, \tag{60}$$

all $\mathbf{A}_{N\lambda_1}$ must be zero. Hence one has a unique decomposition

$$\mathbf{A} = \sum_N \mathbf{A}_{N\lambda_1+2\lambda_2} \tag{61}$$

subject to the Lorentz gauge condition.

Given an electric/magnetic 4-current $\mathbf{J}$, the continuity equation

$$-\nabla_a J^a = * \mathrm{d} * \tilde{\mathbf{J}} = 0 \tag{62}$$

implies that there is the unique decomposition

$$\tilde{\mathbf{J}} = \sum_N \tilde{\mathbf{J}}_{N\lambda_1+2\lambda_2}\,, \tag{63}$$

where $\tilde{\mathbf{J}}_{N\lambda_1+2\lambda_2} \in \Omega^1(dS^4)_{N\lambda_1+2\lambda_2}$.

Now consider the solution

$$\mathbf{F}^{(\mathrm{e})} = \mathrm{d}\mathbf{A}^{(\mathrm{e})} \tag{64}$$

of the Maxwell equations

$$\mathrm{d}\mathbf{F}^{(\mathrm{e})} = 0\,, \quad * \mathrm{d} * \mathbf{F}^{(\mathrm{e})} = -\frac{4\pi}{c}\tilde{\mathbf{J}}^{(\mathrm{e})} \tag{65}$$

in Gaussian units. Having got the unique decompositions

$$\mathbf{J}^{(\mathrm{e})} = \sum_N \mathbf{J}^{(\mathrm{e})}_{N\lambda_1+2\lambda_2}\,, \quad \mathbf{A}^{(\mathrm{e})} = \sum_N \mathbf{A}^{(\mathrm{e})}_{N\lambda_1+2\lambda_2}\,, \tag{66}$$

the Maxwell equations and eq. (50) result in

$$\tilde{\mathbf{J}}^{(\mathrm{e})}_{N\lambda_1+2\lambda_2} = \frac{c}{4\pi}\frac{(N+2)(N+3)}{l^2}\mathbf{A}^{(\mathrm{e})}_{N\lambda_1+2\lambda_2}\,. \tag{67}$$

As for the Maxwell equations with magnetic 4-currents $\mathbf{J}^{(\mathrm{m})}$ only, one has

$$\mathrm{d} * \mathbf{F}^{(\mathrm{m})} = 0\,, \qquad * \mathrm{d}\mathbf{F}^{(\mathrm{m})} = -\frac{4\pi}{c}\tilde{\mathbf{J}}^{(\mathrm{m})}\,. \tag{68}$$

Comparing them with eqs. (65), the solution is rather simple:

$$\mathbf{F}^{(\mathrm{m})} = * \mathrm{d}\mathbf{A}^{(\mathrm{m})} \tag{69}$$

with $\mathbf{A}^{(\mathrm{m})}$ the similar decomposition as $\mathbf{A}$ in eq. (61), provided that the Lorentz gauge condition is also required. Then the Maxwell equations require that

$$\mathbf{J}^{(\mathrm{m})} = \sum_N \mathbf{J}^{(\mathrm{m})}_{N\lambda_1+2\lambda_2}\,, \quad \mathbf{A}^{(\mathrm{m})} = \sum_N \mathbf{A}^{(\mathrm{m})}_{N\lambda_1+2\lambda_2} \tag{70}$$

satisfy

$$\tilde{\mathbf{J}}^{(\mathrm{m})}_{N\lambda_1+2\lambda_2} = \frac{c}{4\pi}\frac{(N+2)(N+3)}{l^2}\mathbf{A}^{(\mathrm{m})}_{N\lambda_1+2\lambda_2}\,. \tag{71}$$

For the Maxwell equations

$$* \,\mathrm{d}\mathbf{F} = -\frac{4\pi}{c}\tilde{\mathbf{J}}_{\mathrm{m}}\,, \qquad * \mathrm{d} * \mathbf{F} = -\frac{4\pi}{c}\tilde{\mathbf{J}}_{\mathrm{e}} \tag{72}$$

with both electric 4-current $\mathbf{J}^{(\mathrm{e})}$ and magnetic 4-current $\mathbf{J}^{(\mathrm{m})}$, we can introduce two potentials $\mathbf{A}^{(\mathrm{e})}$ and $\mathbf{A}^{(\mathrm{m})}$ so that

$$\mathbf{F} = * \mathrm{d}\mathbf{A}^{(\mathrm{m})} + \mathrm{d}\mathbf{A}^{(\mathrm{e})}\,. \tag{73}$$

Then the Maxwell equations force the relations (67) and (71) to be satisfied, provided that the Lorentz gauge condition is satisfied by these potentials, respectively.

So far we have discussed the Maxwell equations by virtue of guage potentials, under the Lorentz gauge condition. By virtue of the direct sum decomposition for both 4-potentials and 4-currents, $\mathbf{A}^{(\mathrm{e})}$ (or $\mathbf{A}^{(\mathrm{m})}$) and $\mathbf{J}^{(\mathrm{e})}$ (or $\mathbf{J}^{(\mathrm{m})}$) can be mutually determined, as shown in eq. (67) (or eq. (71)). It is concluded that there will be no source-free smooth electromagnetic fields in dS^4: if there is neither charge nor current, no matter electric or magnetic, eqs. (67) and (71) will force the corresponding electric/magnetic 4-potential to be zero, which results in a zero electromagnetic field, according to eq. (73).

7 The Mass of Proc Equation

Subject to the Lorentz gauge condition

$$\nabla^a A_a^{(\mathrm{e})} = 0\,, \qquad \nabla^a A_a^{(\mathrm{m})} = 0 \tag{74}$$

as well as eq. (73), the Maxwell equations (72) have the following equivalent form:

$$\nabla_b \nabla^b A_a^{(\mathrm{e})} - R_a{}^b A_b^{(\mathrm{e})} = \frac{4\pi}{c} J_a^{(\mathrm{e})}\,, \qquad \nabla_b \nabla^b A_a^{(\mathrm{m})} - R_a{}^b A_b^{(\mathrm{m})} = \frac{4\pi}{c} J_a^{(\mathrm{m})}\,. \tag{75}$$

The Ricci tensor of dS^4 is $R_{ab} = -\frac{3}{l^2} g_{ab}$. (The negative sign is due to the convention for the metric signature.) Substituting this and eqs. (67), (71) into the above equation, one has

$$\nabla_b \nabla^b A^a_{N\lambda_1+2\lambda_2} - \frac{N^2+5N+3}{l^2} A^a_{N\lambda_1+2\lambda_2} = 0 \tag{76}$$

for both $\mathbf{A}^{(\mathrm{e})}_{N\lambda_1+2\lambda_2}$ and $\mathbf{A}^{(\mathrm{m})}_{N\lambda_1+2\lambda_2}$, for each $N \geqslant 0$. This is a Proca equation with an imaginary mass:

$$m^2 = -\frac{N(N+5)+3}{l^2}\,. \tag{77}$$

Casimir operator.

8 Conclusions and Discussion

Applying the theory of Lie groups and Lie algebras, we have obtained all finite dimensional irreducible $\mathfrak{so}(1,4)$-submodules of $\Omega^1(dS^4)$, the space of differential 1-forms on dS^4. All such $\mathfrak{so}(1,4)$-submodules are Verma modules $\Omega^1(dS^4)_\lambda$, with the highest weight λ being either $N\lambda_1$ or $N\lambda_1 + 2\lambda_2$, where N is a nonnegative integer.

It remains an open problem whether there exists an infinite dimensional irreducible $\mathfrak{so}(1,4)$-submodule of $\Omega^1(dS^4)$. If there is no such an $\mathfrak{so}(1,4)$-submodule in $\Omega^1(dS^4)$, every smooth 1-form on dS^4, $\mathbf{A}$ say, can be decomposed uniquely into a sum as shown in eq. (59). Otherwise an additional term $\mathbf{A}_\infty \in \Omega^1(dS^4)_\infty$ is needed on the right hand side of eq. (59).

As $\mathfrak{so}(1,4)$-modules, $\mathfrak{X}(dS^4)$ (the space of smooth vector fields on dS^4) and $\Omega^3(dS^4)$ are isomorphic to $\Omega^1(dS^4)$. Hence the structures of these three spaces are totally similar.

If $\Omega^1(dS^4)_\infty = 0$, the smooth solution of the Maxwell equations can be directly determined by the smooth electric/magnetic 4-current(s), without neither initial nor boundary conditions. This is as shown in eqs. (67) and (71). An important consequence is that, on dS^4, there is no smooth source-free electromagnetic fields.

Perhaps one would argue that there might be pointed (electric or magnetic) charges so that its electromagnetic fields are still source-free off the position of the charges. Such electromagnetic fields, if exist, is no longer smooth everywhere in the spacetime; and, more importantly, such electromagnetic fields are not source-free, strictly speaking. This problem will be left to discuss in other papers.

Electrodynamics in the Minkowski background tells us that an electromagnetic wave propagates in vacuum at the speed c, carrying together energy-momentum stress tensor. This is how it is accepted that electromagnetic field is also a physical existence, which does not depend on the existence of other matter. Now we have shown that in the de Sitter background it is not the case: the nonexistence of source-free electromagnetic fields means that electromagnetic fields cannot exist independently.

Always companioned by source currents, electromagnetic fields cannot propagate in vacuum in the de Sitter background. Our knowledge of electromagnetism in the Minkowski background raises a question: does the propagation of electromagnetic wave (light) in dS^4 still satisfies the principle of invariant light speed? This is an important question, concerning with the foundation of measurement of the spacetime metric.

Before answering this question, involved concepts must be interpreted. All these and the question itself will be left as a subject in other papers. Here we only present some negative evidence.

The first evidence is that electromagnetic waves often propagate at a lower speed than c in a medium, according to electrodynamics in the Minkowski background. Now that electromagnetic fields cannot exist in vacuum (in the classical sense), it could be strongly suspected that the propagation speed of electromagnetic fields in dS^4 cannot be the "light speed in vacuum".

The second evidence is the potential-current relationship, namely, eqs. (67) and (71): if the electromagnetic wave described by $\mathbf{A}^{(\mathrm{e})}$ or $\mathbf{A}^{(\mathrm{m})}$ "propagates at the light speed in vacuum", then so does the source described by $\mathbf{J}^{(\mathrm{e})}$ or $\mathbf{J}^{(\mathrm{m})}$. However, our knowledge of the special relativity and quantum field theory tells us that charged particles should

not propagate like photons. Although this is based on the Minkowski background, cannot we raise a question if it is violated in the de Sitter background?

The third evidence is that the 4-potential $\mathbf{A}_{N\lambda_1+2\lambda_2} \in \Omega^1(dS^4)_{N\lambda_1+2\lambda_2}$, no matter electric or magnetic, satisfies the Proca equation with a mass such that $m^2 < 0$. Our knowledge based on the Minkowski background reminds us that $\mathbf{A}_{N\lambda_1+2\lambda_2}$ cannot propagate "at the light speed in vacuum".

So far we just present a rough description of our questions, raising by the electromagnetism in the de Sitter background. As have stated, to answer these questions, we should first interpret the precise meaning of these questions. Both the meaning and the answer of these questions concerns seriously with the foundation of physics, especially the foundation of the measurement of spacetime metric in general relativity. Foundation of quantum field theory in curved spacetimes is also concerned with. All these questions will be investigated in further paper.

The conclusions for electromagnetic fields are heavily based on the 4-potentials $\mathbf{A}^{(\mathrm{e})}$ and $\mathbf{A}^{(\mathrm{m})}$. In fact, we can discuss the Maxwell equations without referring to 4-potentials, while the conclusion remains still correct. This will be demonstrated in forthcoming papers.

Background vs Einstein's field equation.

Acknowledgement

A Finite Dimensional Irreducible $\mathfrak{so}(1,4)$-Modules of Smooth Vector Fields on dS^4

B Connection 1-Forms and Curvature 2-Forms

With respect to the coframe

$$\boldsymbol{\theta}^0 = l\,\mathrm{d}\chi\,, \qquad \boldsymbol{\theta}^1 = l\cosh\chi\,\mathrm{d}\zeta\,, \tag{78}$$

$$\boldsymbol{\theta}^2 = l\cosh\chi\cos\zeta\,\mathrm{d}\theta\,, \qquad \boldsymbol{\theta}^3 = l\cosh\chi\sin\zeta\,\mathrm{d}\varphi\,, \tag{79}$$

the connection 1-forms are

$$\boldsymbol{\omega}^0_0 = 0\,, \qquad \boldsymbol{\omega}^0_1 = \frac{1}{l}\tanh\chi\,\boldsymbol{\theta}^1\,, \qquad \boldsymbol{\omega}^0_2 = \frac{1}{l}\tanh\chi\,\boldsymbol{\theta}^2\,, \qquad \boldsymbol{\omega}^0_3 = \frac{1}{l}\tanh\chi\,\boldsymbol{\theta}^3\,, \tag{80}$$

$$\boldsymbol{\omega}^1_0 = \frac{1}{l}\tanh\chi\,\boldsymbol{\theta}^1\,, \qquad \boldsymbol{\omega}^1_1 = 0\,, \qquad \boldsymbol{\omega}^1_2 = \frac{\tan\zeta}{l\cosh\chi}\boldsymbol{\theta}^2\,, \qquad \boldsymbol{\omega}^1_3 = -\frac{\cot\zeta}{l\cosh\chi}\boldsymbol{\theta}^3\,, \tag{81}$$

$$\boldsymbol{\omega}^2_0 = \frac{1}{l}\tanh\chi\,\boldsymbol{\theta}^2\,, \qquad \boldsymbol{\omega}^2_1 = -\frac{\tan\zeta}{l\cosh\chi}\boldsymbol{\theta}^2\,, \qquad \boldsymbol{\omega}^2_2 = 0\,, \qquad \boldsymbol{\omega}^2_3 = 0\,, \tag{82}$$

$$\boldsymbol{\omega}^3_0 = \frac{1}{l}\tanh\chi\,\boldsymbol{\theta}^3\,, \qquad \boldsymbol{\omega}^3_1 = \frac{\cot\zeta}{l\cosh\chi}\boldsymbol{\theta}^3\,, \qquad \boldsymbol{\omega}^3_2 = 0\,, \qquad \boldsymbol{\omega}^3_3 = 0\,. \tag{83}$$

As for the curvature 2-forms $\boldsymbol{\Omega}^\mu_\nu$,

$$\boldsymbol{\Omega}^\mu_\nu = \frac{1}{l^2}\boldsymbol{\theta}^\mu \wedge \boldsymbol{\theta}^\nu\,, \qquad (\text{when } \mu \leqslant \nu)\,; \tag{84}$$

$$\eta_{\mu\rho}\boldsymbol{\Omega}^\rho_\nu + \eta_{\nu\rho}\boldsymbol{\Omega}^\rho_\mu = 0\,. \tag{85}$$

C Self-Dual 2-Forms

Define some local 2-forms

$$\mathbf{S}^1_\pm := \mathrm{d}\chi\wedge\mathrm{d}\zeta \pm i\cosh\chi\sin\zeta\cos\zeta\,\mathrm{d}\theta\wedge\mathrm{d}\varphi\,, \qquad \mathbf{C}^1_\pm = \boldsymbol{\theta}^0\wedge\boldsymbol{\theta}^1 \pm i\,\boldsymbol{\theta}^2\wedge\boldsymbol{\theta}^3 = l^2\cosh\chi\,\mathbf{S}^1_\pm\,, \tag{86}$$

$$\mathbf{S}^2_\pm := \mathrm{d}\chi\wedge\mathrm{d}\theta \mp i\cosh\chi\tan\zeta\,\mathrm{d}\zeta\wedge\mathrm{d}\varphi\,, \qquad \mathbf{C}^2_\pm = \boldsymbol{\theta}^0\wedge\boldsymbol{\theta}^2 \pm i\,\boldsymbol{\theta}^3\wedge\boldsymbol{\theta}^1 = l^2\cosh\chi\cos\zeta\,\mathbf{S}^2_\pm\,, \tag{87}$$

$$\mathbf{S}^3_\pm := \mathrm{d}\chi\wedge\mathrm{d}\varphi \pm i\cosh\chi\cot\zeta\,\mathrm{d}\zeta\wedge\mathrm{d}\theta\,, \qquad \mathbf{C}^3_\pm = \boldsymbol{\theta}^0\wedge\boldsymbol{\theta}^3 \pm i\,\boldsymbol{\theta}^1\wedge\boldsymbol{\theta}^2 = l^2\cosh\chi\sin\zeta\,\mathbf{S}^3_\pm\,. \tag{88}$$

They satisfy

$$*\,\mathbf{S}^j_\pm = \pm i\,\mathbf{S}^j_\pm\,, \qquad *\mathbf{C}^j_\pm = \pm i\,\mathbf{C}^j_\pm\,, \qquad (j=1,2,3) \tag{89}$$

and

$$L_{\mathbf{e}_{\alpha_1}}\mathbf{S}^1_\pm = \frac{i}{2}e^{i\varphi-i\theta}(\mathbf{S}^2_\pm - S^3_\pm)\,, \tag{90}$$

$$L_{\mathbf{e}_{\alpha_1}}\mathbf{S}^2_\pm = -\frac{i}{2}e^{i\varphi-i\theta}(\sec^2\zeta\;\mathbf{S}^1_\pm - i\tan\zeta\;\mathbf{S}^2_\pm + i\tan\zeta\;\mathbf{S}^3_\pm)\,, \tag{91}$$

$$L_{\mathbf{e}_{\alpha_1}}\mathbf{S}^3_\pm = \frac{i}{2}e^{i\varphi-i\theta}(\csc^2\zeta\;\mathbf{S}^1_\pm + i\cot\zeta\;\mathbf{S}^2_\pm - i\cot\zeta\;\mathbf{S}^3_\pm)\,, \tag{92}$$

$$L_{\mathbf{e}_{\alpha_1+2\alpha_2}}\mathbf{S}^1_\pm = -\frac{i}{2}e^{-i\theta-i\varphi}(\mathbf{S}^2_\pm + \mathbf{S}^3_\pm)\,, \tag{93}$$

$$L_{\mathbf{e}_{\alpha_1+2\alpha_2}}\mathbf{S}^2_\pm = \frac{i}{2}e^{-i\theta-i\varphi}(\sec^2\zeta\;\mathbf{S}^1_\pm - i\tan\zeta\;\mathbf{S}^2_\pm - i\tan\zeta\;\mathbf{S}^3_\pm)\,, \tag{94}$$

$$L_{\mathbf{e}_{\alpha_1+2\alpha_2}}\mathbf{S}^3_\pm = \frac{i}{2}e^{-i\theta-i\varphi}(\csc^2\zeta\;\mathbf{S}^1_\pm + i\cot\zeta\;\mathbf{S}^2_\pm + i\cot\zeta\;\mathbf{S}^3_\pm)\,, \tag{95}$$

$$L_{\mathbf{e}_{\alpha_2}}\mathbf{S}^1_\pm = -e^{-i\varphi}\tanh\chi\sin\zeta\;\mathbf{S}^1_\pm \mp e^{-i\varphi}\frac{\cos\zeta}{\cosh\chi}\,\mathbf{S}^2_\pm - ie^{-i\varphi}\tanh\chi\cos\zeta\;\mathbf{S}^3_\pm\,, \tag{96}$$

$$L_{\mathbf{e}_{\alpha_2}}\mathbf{S}^2_\pm = \pm\frac{\sec\zeta}{\cosh\chi}e^{-i\varphi}\mathbf{S}^1_\pm \mp i\frac{\sin\zeta}{\cosh\chi}e^{-i\varphi}\mathbf{S}^3_\pm\,, \tag{97}$$

$$L_{\mathbf{e}_{\alpha_2}}\mathbf{S}^3_\pm = ie^{-i\varphi}\tanh\chi\csc\zeta\cot\zeta\;\mathbf{S}^1_\pm \pm ie^{-i\varphi}\frac{\cos\zeta\cot\zeta}{\cosh\chi}\mathbf{S}^2_\pm - e^{-i\varphi}\tanh\chi\csc\zeta\;\mathbf{S}^3_\pm\,, \tag{98}$$

$$L_{\mathbf{e}_{\alpha_1+\alpha_2}}\mathbf{S}^1_\pm = e^{-i\theta}\tanh\chi\cos\zeta\;\mathbf{S}^1_\pm - ie^{-i\theta}\tanh\chi\sin\zeta\;\mathbf{S}^2_\pm \mp \frac{\sin\zeta}{\cosh\chi}e^{-i\theta}\mathbf{S}^3_\pm\,, \tag{99}$$

$$L_{\mathbf{e}_{\alpha_1+\alpha_2}}\mathbf{S}^2_\pm = ie^{-i\theta}\tanh\chi\sec\zeta\tan\zeta\;\mathbf{S}^1_\pm + e^{-i\theta}\tanh\chi\sec\zeta\;\mathbf{S}^2_\pm \mp ie^{-i\theta}\frac{\sin\zeta\tan\zeta}{\cosh\chi}\mathbf{S}^3_\pm\,, \tag{100}$$

$$L_{\mathbf{e}_{\alpha_1+\alpha_2}}\mathbf{S}^3_\pm = \pm e^{-i\theta}\frac{\csc\zeta}{\cosh\chi}\;\mathbf{S}^1_\pm \pm ie^{-i\theta}\frac{\cos\zeta}{\cosh\chi}\;\mathbf{S}^3_\pm\,. \tag{101}$$

It follows that

$$L_{\mathbf{e}_{\alpha_1}}\mathbf{C}^1_\pm = \frac{i}{2}e^{i\varphi-i\theta}(\sec\zeta\;\mathbf{C}^2_\pm - \csc\zeta\;\mathbf{C}^3_\pm)\,, \qquad L_{\mathbf{e}_{\alpha_1+2\alpha_2}}\mathbf{C}^1_\pm = -\frac{i}{2}e^{-i\theta-i\varphi}(\sec\zeta\;\mathbf{C}^2_\pm + \csc\zeta\;\mathbf{C}^3_\pm)\,, \tag{102}$$

$$L_{\mathbf{e}_{\alpha_1}}\mathbf{C}^2_\pm = -\frac{i}{2}e^{i\varphi-i\theta}(\sec\zeta\;\mathbf{C}^1_\pm + i\,\mathbf{C}^3_\pm)\,, \qquad L_{\mathbf{e}_{\alpha_1+2\alpha_2}}\mathbf{C}^2_\pm = \frac{i}{2}e^{-i\theta-i\varphi}(\sec\zeta\;\mathbf{C}^1_\pm - i\,\mathbf{C}^3_\pm)\,, \tag{103}$$

$$L_{\mathbf{e}_{\alpha_1}}\mathbf{C}^3_\pm = \frac{i}{2}e^{i\varphi-i\theta}(\csc\zeta\;\mathbf{C}^1_\pm + i\,\mathbf{C}^2_\pm)\,, \qquad L_{\mathbf{e}_{\alpha_1+2\alpha_2}}\mathbf{C}^3_\pm = \frac{i}{2}e^{-i\theta-i\varphi}(\csc\zeta\;\mathbf{C}^1_\pm + i\,\mathbf{C}^2_\pm)\,, \tag{104}$$

$$L_{\mathbf{e}_{\alpha_2}}\mathbf{C}^1_\pm = \mp\frac{e^{-i\varphi}}{\cosh\chi}\mathbf{C}^2_\pm - ie^{-i\varphi}\tanh\chi\cot\zeta\;\mathbf{C}^3_\pm\,, \qquad L_{\mathbf{e}_{\alpha_1+\alpha_2}}\mathbf{C}^1_\pm = -ie^{-i\theta}\tanh\chi\tan\zeta\;\mathbf{C}^2_\pm \mp \frac{e^{-i\theta}}{\cosh\chi}\mathbf{C}^3_\pm\,, \tag{105}$$

$$L_{\mathbf{e}_{\alpha_2}}\mathbf{C}^2_\pm = \pm\frac{e^{-i\varphi}}{\cosh\chi}\mathbf{C}^1_\pm \mp i\frac{e^{-i\varphi}\cos\zeta}{\cosh\chi}\mathbf{C}^3_\pm\,, \qquad L_{\mathbf{e}_{\alpha_1+\alpha_2}}\mathbf{C}^2_\pm = ie^{-i\theta}\tanh\chi\tan\zeta\;\mathbf{C}^1_\pm \mp i\frac{e^{-i\theta}\sin\zeta}{\cosh\chi}\mathbf{C}^3_\pm\,, \tag{106}$$

$$L_{\mathbf{e}_{\alpha_2}}\mathbf{C}^3_\pm = ie^{-i\varphi}\tanh\chi\cot\zeta\;\mathbf{C}^1_\pm \pm i\frac{e^{-i\varphi}\cos\zeta}{\cosh\chi}\mathbf{C}^2_\pm\,, \qquad L_{\mathbf{e}_{\alpha_1+\alpha_2}}\mathbf{C}^3_\pm = \pm\frac{e^{-i\theta}}{\cosh\chi}\mathbf{C}^1_\pm \pm i\frac{e^{-i\theta}\cos\zeta}{\cosh\chi}\mathbf{C}^3_\pm\,. \tag{107}$$

References

[1] B. Zhou and Z.-H. Zhou, "Smooth Klein-Gordon Scalars in the de Sitter Background: Massless or Tachyonic Only", unpublished.

[2]

[3]

[4] Donato Bini, Salvatore Capozziello and Giampiero Esposito, "Gravitational waves about curved backgrounds: a consistency analysis in de Sitter spacetime", *Int. J. Geom. Meth. Mod. Phys.* **5** (2008), 1069–1083.

[5] Donato Bini, Giampiero Esposito, Roberto Valentino Montaquila, "Solution of Maxwell's equations on a de Sitter background", *Gen. Rel. Grav.* **42** (2010), 51–61, (arXiv:0812.1973).

[6] J. M. Cohen and L. S. Kegeles, *Phys. Rev.* **D 10** (1974), 1070.